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RESEARCH ARTICLE

Identities and quasiidentities in the lattice of overcommutative semigroup varieties

V.Y. Shaprynski

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Abstract We describe overcommutative varieties of semigroups whose lattice of overcommutative subvarieties satisfies a non-trivial identity or quasiidentity. These two properties turn out to be equivalent.

Keywords Semigroup · Variety · Overcommutative variety · Subvariety lattice · Lattice identity · Lattice quasiidentity

1 Introduction and summary

It is generally known that the lattice of all semigroup varieties is a disjoint union of two wide and important sublattices: the ideal of all periodic varieties and the co-ideal of all *overcommutative* varieties, that is, varieties containing the variety $\mathcal{COM}$ of all commutative semigroups. We denote the lattice of all overcommutative varieties by **OC**.

By $\mathrm{L}(\mathcal{V})$ we denote the subvariety lattice of a semigroup variety $\mathcal{V}$. Identities and quasiidentities in lattices $\mathrm{L}(\mathcal{V})$ were investigated in several papers, see Sects. 11 and 12 in the survey [8]. The results of [2] and [7] imply that no non-trivial lattice quasiidentity holds in the lattice of commutative semigroup varieties and hence in the lattice $\mathrm{L}(\mathcal{V})$ whenever $\mathcal{V}$ is overcommutative. Therefore investigation of identities and quasiidentities in lattices $\mathrm{L}(\mathcal{V})$ gives no information about the lattice **OC**. In view of

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V.Y. Shaprynski (✉)
Institute of Mathematics and Computer Science, Ural Federal University, Lenina 51, 620083 Ekaterinburg, Russia
e-mail: vshapr@yandex.ru

this fact it is natural to study identities and quasiidentities in lattices of overcommutative subvarieties of overcommutative varieties. For an overcommutative variety $\mathcal{V}$, its lattice of overcommutative subvarieties (that is, the interval between $\mathcal{COM}$ and $\mathcal{V}$) will be denoted by $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$.

The structure of the lattice **OC** has been revealed by Volkov in [11]. We shall give the formulations of the results of this paper in Sect. 2. Basing on the results of [11], Vernikov described overcommutative varieties whose lattice of overcommutative subvarieties is distributive, modular, Arguesian, lower or upper semimodular, lower or upper semidistributive or satisfies some other related restrictions [9, 10]. In the present paper we describe overcommutative varieties $\mathcal{V}$ whose lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ satisfies a non-trivial lattice identity or quasiidentity.

We need the following definitions and notation. Lattices are called [*quasi*]*equationally* equivalent if they satisfy the same [quasi]identities. A semigroup variety $\mathcal{V}$ is *permutative* if it satisfies an identity of the form

$$x_1x_2\cdots x_n = x_{g(1)}x_{g(2)}\cdots x_{g(n)} \tag{1}$$

where g is a non-trivial permutation on the set $\{1,\ldots,n\}$. The semigroup variety given by an identity system Σ is denoted by $\operatorname{var}\Sigma$. Put

$$\mathcal{LZ} = \operatorname{var}\{xy = x\}, \qquad \mathcal{RZ} = \operatorname{var}\{xy = y\},$$
$$\mathcal{X} = \operatorname{var}\{xyzt = xytz,\ x^2y^2 = y^2x^2 = (xy)^2\}.$$

The variety dual to $\mathcal{X}$ is denoted by $\overleftarrow{\mathcal{X}}$.

The main result of this article is

Theorem 1.1 *For an overcommutative semigroup variety $\mathcal{V}$, the following are equivalent*:

(a) *the lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ satisfies a non-trivial lattice identity*;
(b) *the lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ satisfies a non-trivial lattice quasiidentity*;
(c) *the lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ is equationally equivalent to a finite lattice*;
(d) *the lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ is quasiequationally equivalent to a finite lattice*;
(e) *the variety $\mathcal{V}$ is permutative and contains none of the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{X}$, $\overleftarrow{\mathcal{X}}$.*

Since every finite lattice has a finite identity basis [4], Theorem 1.1 immediately imply the following

Corollary 1.2 *If $\mathcal{V}$ is an overcommutative variety and the lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ satisfies a non-trivial identity then this lattice has a finite identity basis.*

The article consists of four sections. Sections 2 and 3 contain preliminary results. In Sect. 4 the proof of Theorem 1.1 is given.

2 Subdirect decomposition of the lattice OC

The aim of this section is to formulate the results of [11]. In order to do this, we need some new definitions and notation. The free semigroup over the countably infinite alphabet $X = \{x_1, x_2, \dots\}$ is denoted by F. The symbol $\equiv$ stands for the equality relation on F. Put $X_m = \{x_1, \dots, x_m\}$. Let F_m be the free semigroup over the set X_m. If w is a word then we denote the length of w by $\ell(w)$ and the number of occurrences of a letter x_i in w by $\ell_{x_i}(w)$ or, shortly, by $\ell_i(w)$. The symmetric group on the set $\{1, \dots, m\}$ is denoted by $\mathbb{S}_m$. For $g \in \mathbb{S}_m$ and $1 \le i \le m$, we put $g(x_i) = x_{g(i)}$ thus identifying $\mathbb{S}_m$ with the symmetric group on X_m. The lattice of all equivalence relations on a set A is denoted by $\mathrm{Part}(A)$. If a group G acts on A then we say that A is a *G-set* and regard A as a unary algebra with the set of operations G. This allows us to consider congruences of G-sets. The congruence lattice of a G-set A is denoted by $\mathrm{Con}(A)$. If L is a lattice and $x \in L$ then $(x]$ (respectively, $[x)$) stands for the principal ideal (respectively, co-ideal) generated by the element x. By $\overline{L}$ we denote the dual lattice to a lattice L.

A *partition* is a sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_m)$ where $\lambda_1 \ge \dots \ge \lambda_m$ and $m \ge 2$. The set of all partitions is denoted by Λ. Let us fix a partition λ. We say that λ is a *partition of the number n into m parts* where $n = \sum_{i=1}^{m} \lambda_i$. The numbers λ_i are called *components* of λ. We consider the set

$$W_\lambda = \left\{ w \in F_m \mid \ell_i(w) = \lambda_i \text{ for } 1 \le i \le m \right\}$$

and the group

$$G_\lambda = \left\{ g \in \mathbb{S}_m \mid \lambda_i = \lambda_{g(i)} \text{ for } 1 \le i \le m \right\}.$$

Every element $g \in G_\lambda$, as a permutation on the alphabet X_m, defines a permutation on the set W_λ which renames letters in each word in W_λ. This means that the group G_λ acts on the set W_λ and this set is considered as a G_λ-set. For an overcommutative variety $\mathcal{V}$, we define an equivalence relation $\varphi_\lambda(\mathcal{V})$ on W_λ as the restriction to the set W_λ of the fully invariant congruence on F corresponding to $\mathcal{V}$. Thus a mapping $\varphi_\lambda \colon \mathbf{OC} \to \mathrm{Part}(W_\lambda)$ is defined.

Proposition 2.1 [11] *Every mapping φ_λ is a homomorphism of the lattice* **OC** *onto the lattice $\overline{\mathrm{Con}(W_\lambda)}$. These homomorphisms are components of an embedding*

$$\varphi = (\varphi_\lambda)_{\lambda \in \Lambda} \colon \mathbf{OC} \longrightarrow \prod_{\lambda \in \Lambda} \overline{\mathrm{Con}(W_\lambda)}$$

which decomposes the lattice **OC** *into a subdirect product of the lattices $\overline{\mathrm{Con}(W_\lambda)}$, $\lambda \in \Lambda$.*

One can generalize Proposition 2.1 in order to obtain a subdirect decomposition of the lattice $\mathrm{L_{OC}}(\mathcal{V})$. As a surjective homomorphism, φ_λ maps principal ideals to principal ideals, so

$$\varphi_\lambda\big(\mathrm{L_{OC}}(\mathcal{V})\big) = \big(\varphi_\lambda(\mathcal{V})\big]_{\overline{\mathrm{Con}(W_\lambda)}} = \overline{\big[\varphi_\lambda(\mathcal{V})\big)_{\mathrm{Con}(W_\lambda)}}.$$

The co-ideal $[\varphi_\lambda(\mathcal{V}))_{\mathrm{Con}(W_\lambda)}$ is isomorphic to the congruence lattice of the quotient G_λ-set $W_\lambda/\varphi_\lambda(\mathcal{V})$. Thus we have

Corollary 2.2 [11] *For any variety $\mathcal{V} \in \mathbf{OC}$, the homomorphism $\varphi|_{\mathrm{L}_{\mathbf{OC}}(\mathcal{V})}$ defines a decomposition of the lattice $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$ into a subdirect product of the lattices $\overline{\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))}$.*

Another result we need is

Proposition 2.3 [11] *Every lattice $\overline{\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))}$ can be embedded into $\mathrm{L}_{\mathbf{OC}}(\mathcal{V})$.*

3 Preliminaries on semigroup identities

In this section we study some equational properties of the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{X}$ and $\overleftarrow{\mathcal{X}}$. The following two lemmas and their duals give the solution to the word problem in these varieties. For the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ it is generally known and evident.

Lemma 3.1 *An identity $u = v$ holds in $\mathcal{LZ}$ if and only if the words u and v start with the same letters.*

An identity $u = v$ is called *balanced* if $\ell_x(u) = \ell_x(v)$ for every $x \in X$. All identities satisfied by overcommutative varieties are balanced. A letter x in a word $w \in F$ is called *simple* if $\ell_x(w) = 1$ and *multiple* otherwise.

Lemma 3.2 *An identity $u = v$ holds in $\mathcal{X}$ if and only if it is balanced and at least one of the following holds*:

(i) $u \equiv v \in X$;
(ii) *u and v have equal first letters and equal second letters*;
(iii) *u and v have equal first letters and their second letters are multiple*;
(iv) *the first and the second letters in u and v are multiple.*

Proof Let us denote by α the fully invariant congruence on F corresponding to $\mathcal{X}$ and by β the set of all balanced identities $u = v$ (considered as pairs of words) satisfying one of the conditions (i)–(iv). We must prove that $\alpha = \beta$.

First, one can prove that $\alpha \subseteq \beta$. The identity $xyzt = xytz$ satisfies (ii) while the identities $x^2y^2 = y^2x^2 = (xy)^2$ satisfy (iv), so all these identities belong to β. A straightforward verification shows that β is a fully invariant congruence on F. This implies the desired inclusion.

It remains to verify that $\beta \subseteq \alpha$. We shall prove that a balanced identity $u = v$ holds in $\mathcal{X}$ in each of the cases (i)–(iv). The case (i) is trivial. The identity $xyzt = xytz$ implies every identity of the kind (1) with $g(1) = 1$ and $g(2) = 2$. Identifying and renaming letters in the latter identity, one can obtain every identity with the property (ii). In the rest of the proof we suppose that the identity $u = v$ is written in the form $xya = ztb$ where $x, y, z, t \in X$ and $a, b \in F$ (the letters x, y, z, and t are not assumed to be distinct). Consider the case (iv). Suppose that $x \equiv z \equiv t$ and $x \not\equiv y$, that is $u \equiv xya$ and $v \equiv x^2b$. Since y is multiple, there exist balanced identities of the form $xya = (xy)^2c$ and $x^2b = x^2y^2c$ for some $c \in F$. These identities satisfy (ii), so they hold in $\mathcal{X}$. Hence we have

$$xya = (xy)^2c = x^2y^2c = x^2b$$

in $\mathcal{X}$. The same arguments show that $\mathcal{X}$ satisfies $u = v$ whenever $y \equiv z \equiv t$ and $x \not\equiv y$ (one should use the identity $(xy)^2 = y^2x^2$ rather than $(xy)^2 = x^2y^2$ in this case). Therefore in the general case $\mathcal{X}$ satisfies

$$xya = x^2c = xtd = t^2e = ztb \quad \text{where } c, d, e \in F$$

whenever these identities are balanced. Of course, such words c, d, and e exist, so we are done in the case (iv). In the case (iii) the identity $u = v$ is $xya = xtb$ where y and t are multiple. We may suppose that the letter x is simple, because otherwise the property (iv) holds. In particular, $x \not\equiv y$ and $x \not\equiv z$. The variety $\mathcal{X}$ satisfies $xya = xy^2c$ and $xtb = xt^2d$ $(c, d \in F)$ whenever these identities are balanced (the case (ii)). Furthermore, $\mathcal{X}$ satisfies $y^2c = z^2d$ (the case (iv)), so it satisfies $xya = xy^2c = xt^2d = xtb$. □

For a non-negative integer k, consider the variety

$$\mathcal{P}_k = \mathrm{var}\{x_1 \cdots x_k yzt_1 \cdots t_k = x_1 \cdots x_k zyt_1 \cdots t_k\}.$$

This variety satisfies every balanced identity of the form $acb = adb$ where $\ell(a) = \ell(b) = k$.

Lemma 3.3 [6] *Every permutative variety is contained in $\mathcal{P}_k$ for some k.*

Lemma 3.4 *Every overcommutative permutative variety $\mathcal{V}$ such that $\mathcal{LZ} \nsubseteq \mathcal{V}$ satisfies the identity*

$$x^ny^nz^n = y^nx^nz^n \tag{2}$$

for any sufficiently large n.

Proof Being permutative, the variety $\mathcal{V}$ is contained in $\mathcal{P}_k$ for some k by Lemma 3.3. Lemma 3.1 and the fact that $\mathcal{LZ} \nsubseteq \mathcal{V}$ implies that the variety $\mathcal{V}$ satisfies an identity $xa = yb$ where $x \not\equiv y$. The identity $xa = yb$ is balanced because $\mathcal{V}$ is overcommutative. We may suppose that a and b contain only the letters x and y. If this is not the case then we identify all other letters with x. Assume that $n \geq k + \ell(a) = k + \ell(b)$. We are going to prove that $\mathcal{V}$ satisfies all identities of the form $cz^n = dz^n$ where c and d contain only the letters x and y and $\ell_x(c) = \ell_x(d) = \ell_y(c) = \ell_y(d) = n$. Take the greatest common prefix e of the words c and d. There are words c' and d' with $c \equiv exc'$ and $d \equiv eyd'$. If $\ell(e) \geq k$ then the identity

$$cz^n \equiv exc'z^n = eyd'z^n \equiv dz^n$$

holds in $\mathcal{V}$ because $\mathcal{V} \subseteq \mathcal{P}_k$ and $n > k$. Suppose that $0 \le \ell(e) \le k$. To prove that $\mathcal{V}$ satisfies $cz^n = dz^n$ in this case, we use inverse induction by $\ell(e)$. As the induction base we take the case $\ell(e) = k$ which has already been considered. Now we shall prove the statement for $\ell(e) < k$ assuming that it is proved for greater $\ell(e)$. Put $p = \ell_x(e) + \ell_x(b)$ and $q = \ell_y(e) + \ell_y(a)$. The inequality $n \ge k + \ell(a) = k + \ell(b)$ imply $n > p$ and $n > q$. The variety $\mathcal{V}$ satisfies

$$\begin{aligned} cz^n \equiv exc'z^n &= exax^{n-p}y^{n-q}z^n && \text{by the induction assumption} \\ &= eybx^{n-p}y^{n-q}z^n && \text{because } xa = yb \\ &= eyd'z^n \equiv dz^n && \text{by the induction assumption,} \end{aligned}$$

as was to be proved. □

Lemma 3.5 *Every overcommutative permutative variety $\mathcal{V}$ such that $\mathcal{LZ}, \mathcal{X} \nsubseteq \mathcal{V}$ satisfies the identity*

$$xtx^{n-1}y^nz^n = yty^{n-1}x^nz^n \tag{3}$$

for any sufficiently large n.

Proof By Lemma 3.3 we have $\mathcal{V} \subseteq \mathcal{P}_k$ for some k. By Lemma 3.4 the variety $\mathcal{V}$ satisfies

$$x^my^mz^m = y^mx^mz^m \tag{4}$$

for some $m \ge k$. The variety $\mathcal{V}$ satisfies a balanced identity $u = v$ which fails in $\mathcal{X}$. According to Lemma 3.2, there are four possible cases.

Case 1. The first letters in u and v coincide, the second letters are distinct and at least one of the second letters is simple. Identifying all letters in $u = v$ except this simple letter, we obtain an identity of the form

$$xyx^{p+q-1} = x^{p+1}yx^{q-1} \tag{5}$$

for some p and q. This identity implies $xyx^{pr+q-1} = x^{pr+1}yx^{q-1}$ for all positive integers r, so p can be replaced by pr in (5). This allows us to suppose that $p \ge k$. Let us take n with $n \ge m + k$ and $n \ge p + q$. The variety $\mathcal{V}$ satisfies

$$\begin{aligned} xtx^{n-1}y^nz^n &= x^{p+1}tx^{n-p-1}y^nz^n && \text{by (5)} \\ &= x^my^mz^mtx^{n-m}y^{n-m}z^{n-m} && \text{because } \mathcal{V} \subseteq \mathcal{P}_k \\ &= y^mx^mz^mtx^{n-m}y^{n-m}z^{n-m} && \text{by (4)} \\ &= y^{p+1}ty^{n-p-1}x^nz^n && \text{because } \mathcal{V} \subseteq \mathcal{P}_k \\ &= yty^{n-1}x^nz^n && \text{by (5).} \end{aligned}$$

Case 2. The first letters in u and v are distinct and at least one of these letters is simple. Identifying all letters in $u = v$ except this simple letter we obtain

$yx^{p+q} = x^p yx^q$ for some positive p and non-negative q. This identity implies $xyx^{p+q} = x^{p+1}yx^q$, so we return to the Case 1.

Case 3. The second letters in u and v coincide and are simple while the first letters are distinct and multiple. Let us write the identity $u = v$ in the form $xtu' = ytv'$. We may suppose that u' and v' contain only the letters x and y because all other letters can be identified with x. Put $p = \ell_x(v')$ and $q = \ell_y(u')$. Let us take n with $n \ge k + p$, $n \ge q$, $n \ge k + m$, and $n \ge m + 1$. We have that

$$\begin{aligned}
xtx^{n-1}y^n z^n &= xtx^k u' x^{n-k-p} y^{n-q} z^n && \text{because } \mathcal{V} \subseteq \mathcal{P}_k \\
&= ytx^k v' x^{n-k-p} y^{n-q} z^n && \text{because } xtu' = ytv' \\
&= ytx^m y^m z^m x^{n-m} y^{n-m-1} z^{n-m} && \text{because } \mathcal{V} \subseteq \mathcal{P}_k \\
&= yty^m x^m z^m x^{n-m} y^{n-m-1} z^{n-m} && \text{by (4)} \\
&= yty^{n-1} x^n z^n && \text{because } \mathcal{V} \subseteq \mathcal{P}_k
\end{aligned}$$

holds in the variety $\mathcal{V}$.

Case 4. The first letters in u and v are distinct and multiple, the second letters are distinct, and at least one of the second letters is simple. Identifying the first letters in the words u and v, we return to the Case 1. □

4 Proof of Theorem 1.1

The proof follows the scheme (a) → (b) → (e) → (d) → (c) → (a). The implications (a) → (b) and (d) → (c) are obvious. The implication (c) → (a) holds because every finite lattice satisfies a non-trivial identity (see [3, Lemma V.3.2], for instance). It remains to verify the implications (b) → (e) → (d).

(b) → (e) Arguing by contradiction, suppose that the property (e) fails. We shall prove that every finite lattice can be embedded into one of the lattices $\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))$. Hence every finite lattice can be embedded into $\mathbf{L_{OC}}(\mathcal{V})$ by Proposition 2.3. Since every non-trivial lattice quasiidentity fails in some finite lattice [1], this will give us the contradiction we need. There are three cases to consider.

Case 1. The variety $\mathcal{V}$ is not permutative. Consider the partition $\lambda = (\underbrace{1, \ldots, 1}_{n \text{ times}})$. For this partition we have $G_\lambda = \mathbb{S}_n$. The corresponding G_λ-set W_λ is regular (i. e., it is transitive and any non-unit element of G_λ has no fixed points). In this case $\mathrm{Con}(W_\lambda) \cong \mathrm{Sub}(G_\lambda) = \mathrm{Sub}(\mathbb{S}_n)$ where $\mathrm{Sub}(G)$ is the subgroup lattice of a group G (see [5, Lemma 4.20]). Since the variety $\mathcal{V}$ is not permutative, the congruence $\varphi_\lambda(\mathcal{V})$ is the equality relation on W_λ, so $W_\lambda/\varphi_\lambda(\mathcal{V}) = W_\lambda$. We have obtained that $\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V})) \cong \mathrm{Sub}(\mathbb{S}_n)$. Every finite lattice can be embedded into a lattice $\mathrm{Sub}(\mathbb{S}_n)$ for some n [7], so we are done.

Case 2. The variety $\mathcal{V}$ contains one of the subvarieties $\mathcal{LZ}$ and $\mathcal{RZ}$. By duality principle, we may suppose that $\mathcal{LZ} \subseteq \mathcal{V}$. Consider the partition $\lambda =$

$(m, m-1, \dots, 2, 1)$ for an arbitrary $m \geq 2$. The group G_λ is trivial, whence $\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V})) = \mathrm{Part}(W_\lambda/\varphi_\lambda(\mathcal{V}))$. Since $\mathcal{LZ} \subseteq \mathcal{V}$, the variety $\mathcal{V}$ satisfies no identity $u = v$ where the first letters in u and v are distinct. In particular, $(x_i a, x_j b) \notin \varphi_\lambda(\mathcal{V})$ whenever $x_i a, x_j b \in W_\lambda$ and $i \neq j$. Hence the set $W_\lambda/\varphi_\lambda(\mathcal{V})$ contains at least m elements. Any finite lattice can be embedded into any sufficiently large finite partition lattice [7], so it can be embedded into some of the lattices $\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))$.

Case 3. The variety $\mathcal{V}$ contains one of the subvarieties $\mathcal{X}$ and $\overleftarrow{\mathcal{X}}$, say, $\mathcal{X} \subseteq \mathcal{V}$. Consider the same partition λ as in Case 2. Again we have $\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V})) = \mathrm{Part}(W_\lambda/\varphi_\lambda(\mathcal{V}))$. Since $\mathcal{X} \subseteq \mathcal{V}$, Lemma 3.2 implies that the variety $\mathcal{V}$ satisfies no identity $u = v$ where the first letters in u and v are distinct and the second letter in u is simple. In particular, $(x_i x_m a, x_j x_m b) \notin \varphi_\lambda(\mathcal{V})$ whenever $x_i x_m a, x_j x_m b \in W_\lambda$ and $i \neq j$. Hence the set $W_\lambda/\varphi_\lambda(\mathcal{V})$ contains at least $m-1$ elements, so we are done, as in Case 2.

(e) $\rightarrow$ (d). Let $\mathcal{V}$ be an overcommutative variety satisfying (e). Consider the subdirect decomposition of the lattice $\mathbf{L_{OC}}(\mathcal{V})$ given by Corollary 2.2. We will prove that the cardinalities of the subdirect factors $\overline{\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))}$ are bounded. This implies that there exists only a finite number of non-isomorphic lattices among these factors. The lattice $\mathbf{L_{OC}}(\mathcal{V})$ is quasiequationally equivalent to the direct product of these distinct factors because quasiidentities are preserved under taking sublattices and direct products. Therefore the implication will be proved.

Let us fix a partition λ. The variety $\mathcal{V}$ is contained in $\mathcal{P}_k$ for some k by Lemma 3.3. We may assume that λ is a partition of a number greater than $2k+1$. Indeed, there is only a finite number of other partitions and existence of an upper bound for $|\mathrm{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))|$ does not depend on them. By Lemmas 3.4, 3.5 and their duals the variety $\mathcal{V}$ satisfies the identities (2), (3), and their duals for some n. Consider the set I of all integers i, $1 \leq i < n+2k$, such that at least $4k$ components of λ are equal to i. For every $i \in I$, we fix a set of letters Y_i such that $|Y_i| = 4k$ and $\lambda_j = i$ whenever $x_j \in Y_i$. This means that $\ell_x(w) = i$ whenever $x \in Y_i$ and $w \in W_\lambda$. Each word $w \in W_\lambda$ can be written as $w \equiv abc$ where $\ell(a) = \ell(c) = k$. We denote a, b, and c by $L(w)$, $M(w)$, and $R(w)$, respectively. Note that $(w_1, w_2) \in \varphi_\lambda(\mathcal{V})$ whenever $w_1, w_2 \in W_\lambda$, $L(w_1) = L(w_2)$, and $R(w_1) = R(w_2)$. Consider the following two restrictions on a word $w \in W_\lambda$:

(i) there are no letters x in the words $L(w)$ and $R(w)$ with $\ell_x(w) \geq n+2k$ and $x \not\equiv x_1$, $x \not\equiv x_2$ (recall that $\ell_1(w) \geq \ell_2(w) \geq \ell_x(w)$ for any $x \in X \setminus \{x_1, x_2\}$, so this property trivially holds whenever $\ell_2(w) < n+2k$);
(ii) there are no letters x in the words $L(w)$ and $R(w)$ with $\ell_x(w) = i \in I$ and $x \notin Y_i$.

Let us prove that, for any $w \in W_\lambda$, there exist $w' \in W_\lambda$ with the property (i) and such that $w = w'$ in $\mathcal{V}$. This means that each $\varphi_\lambda(\mathcal{V})$-class contains a word with the property (i). Consider an occurrence in $L(w)$ of a letter x with $\ell_x(w) > n+2k$, $x \not\equiv x_1$, and $x \not\equiv x_2$. There are words d and e with $L(w) \equiv dxe$. Since $\ell_1(w) \geq \ell_2(w) \geq \ell_x(w) \geq n+2k$, we have

$$\ell_x\bigl(M(w)\bigr), \ell_1\bigl(M(w)\bigr), \ell_2\bigl(M(w)\bigr) \geq n.$$

Hence there exists a balanced identity of the form $M(w) = x^{n-1}x_1^n x_2^n f$ for some word f. The variety $\mathcal{V}$ satisfies

$$\begin{aligned} w &\equiv L(w)M(w)R(w) \\ &\equiv dxeM(w)R(w) \\ &= dxex^{n-1}x_1^n x_2^n fR(w) \quad \text{because } \mathcal{V} \subseteq \mathcal{P}_k \\ &= dx_1ex_1^{n-1}x^n x_2^n fR(w) \quad \text{by (2) if } e \text{ is empty or by (3) otherwise.} \end{aligned}$$

The word $w'' \equiv dx_1ex_1^{n-1}x^n x_2^n fR(w)$ is such that $L(w'') \equiv dx_1e$, $R(w'') \equiv R(w)$, and $(w, w'') \in \varphi_\lambda(\mathcal{V})$. We have excluded one occurrence of the letter x in $L(w)$. Repeating this procedure one can exclude all occurrences in $L(w)$ of letters x with $\ell_x(w) > n + 2k$ except x_1 and x_2. Dually, one can exclude all occurrences of such letters in $R(w)$.

Now we shall prove that every identity $u = v$ such that $u, v \in W_\lambda$ is equivalent to an identity $u' = v'$ where u' and v' satisfy (ii). Since

$$\ell\bigl(L(u)\bigr) + \ell\bigl(L(v)\bigr) + \ell\bigl(R(u)\bigr) + \ell\bigl(R(v)\bigr) = 4k,$$

the words $L(u)$, $L(v)$, $R(u)$, and $R(v)$ contain at most $4k$ distinct letters. Therefore, for $1 \le i < n + 2k$, they contain at most $4k$ distinct letters x with $\ell_x(u) = i$. Consider any element $g \in G_\lambda$ which maps, for every $1 \le i < n + 2k$, all letters x in $L(u), L(v), R(u), R(v)$ with $\ell_x(w) = i$ to the set Y_i. To obtain the identity $u' = v'$, one may take $u' \equiv g(u)$ and $v' \equiv g(v)$.

Combining the statements in the previous two paragraphs, we conclude that every identity $u = v$ with $u, v \in W_\lambda$ is equivalent within the variety $\mathcal{V}$ to an identity $u' = v'$ where u' and v' satisfy (i) and (ii). This statement may be reformulated in terms of G-sets. To do this, denote by A the set of $\varphi_\lambda(\mathcal{V})$-classes of all words in W_λ satisfying (i) and (ii). We have proved that every congruence on $W_\lambda/\varphi_\lambda(\mathcal{V})$ is generated by some subset of $A \times A$. The $\varphi_\lambda(\mathcal{V})$-class of w is defined by $L(w)$ and $R(w)$ and does not depend on $M(w)$. Conditions (i) and (ii) mean that $L(w)$ and $R(w)$ for all such w may contain at most $4k(2n+k-1)+2$ distinct letters in common: at most $4k$ letters x with $\ell_x(w) = i$ for every $1 \le i < 2n + k$ and at most 2 letters x with $\ell_x(w) \ge 2n + k$. Hence $|A| \le N$ where $N = (4k(2n + k - 1) + 2)^{2k}$. Therefore

$$\bigl|\operatorname{Con}\bigl(W_\lambda/\varphi_\lambda(\mathcal{V})\bigr)\bigr| \le 2^{|A\times A|} \le 2^{N^2}.$$

This upper bound does not depend on the partition λ.

Theorem 1.1 is proved.

Remark 4.1 The proof of the implication (e) $\to$ (d) is based on the fact that the cardinalities of the lattices $\operatorname{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))$ are bounded whenever $\mathcal{V}$ satisfies (e). However the cardinalities of the sets $W_\lambda/\varphi_\lambda(\mathcal{V})$ can be unbounded. For example, put

$$\mathcal{V} = \operatorname{var}\bigl\{x^2y = yx^2,\ xyz = xzy\bigr\}.$$

For the partition $\lambda = (\underbrace{1, \dots, 1}_{n \text{ times}})$, it is easy to verify that the set $W_\lambda/\varphi_\lambda(\mathcal{V})$ contains exactly n elements.

Remark 4.2 The variety $\mathcal{LZ}$ is well-known to be an atom of the lattice of all semigroup varieties. But the lattice $\mathbf{L_{OC}}(\mathcal{COM} \vee \mathcal{LZ})$ is not small as it might has been expected. The proof of Theorem 1.1 shows that this lattice contains an isomorphic copy of every finite lattice (see Case 2 in the proof of the implication (b) $\to$ (e)).

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